

Volume preserving multidimensional integrable systems and Nambu-Poisson Geometry

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dedicated to the memory of Dr. B.C. Guha

Abstract

In this paper we study generalized classes of volume preserving multidimensional integrable systems via Nambu-Poisson mechanics. These integrable systems belong to the same class of dispersionless KP type equation. Hence they bear a close resemblance to the self dual Einstein equation. Recently Takasaki-Takebe provided the twistor construction of dispersionless KP and dToda type equations by using the Gindikin's pencil of two forms. In this paper we generalize this twistor construction to our systems.

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1 Introduction

In this article we study volume preserving diffeomorphic integrable hierarchy of three flows [1]. This is different from the usual two flows cases and this can be studied via Nambu-Poisson geometry. It is already known that a group of volume preserving diffeomorphisms in three dimension plays a key role in an Einstein-Maxwell theory whose Weyl curvature is self-dual and whose Maxwell tensor has an algebraically anti self-dual. Later Takasaki [2] explicitly showed how volume preserving diffeomorphisms arises in integrable deformations of self-dual gravity.

Nambu mechanics is a generalization of classical Hamiltonian mechanics, introduced by Yoichiro Nambu [3]. At the begining he wanted to formulate a statistical mechanics on \mathbf{R}^3 , emphasizing that the only feature of Hamiltonian mechanics one should preserve is the Liouville theorem. He considered the following equations of motion

$$\frac{d\mathbf{r}}{dt} = \nabla u(\mathbf{r}) \wedge \nabla v(\mathbf{r}), \quad \mathbf{r} = (x, y, z) \in \mathbf{R},$$

where x, y, z are dynamical variables and u, v are two functions of \mathbf{r} . Then Liouville theorem follows from the identity

$$\nabla \cdot (\nabla u(\mathbf{r}) \wedge \nabla v(\mathbf{r})) = 0.$$

He further observed that the above equation of motion can be cast into

$$\frac{df}{dt} = \frac{\partial(f, u, v)}{\partial(x, y, z)},$$

the Jacobian of right hand side can be interpreted as a generalized Poisson bracket.

Hence the binary operation of Poisson bracket of Hamiltonian mechanics is generalized to n -ary operation in Nambu mechanics. Recently Takhtajan [4,5] has formulated its basic principles in an invariant geometrical form similar to that of Hamiltonian mechanics.

In this paper we shall use Nambu mechanics to study generalized volume preserving diffeomorphic integrable hierarchy. These classes of integrable systems are closely related to the self dual Einstein equation, dispersionless KP equations etc. In fact we obtain a higher dimensional analogue of all these systems. It turns out that all these systems can be written in the following form:

$$\begin{aligned} d\Omega^{(n)} &= 0 \\ \Omega^{(n)} \wedge \Omega^{(n)} &= 0. \end{aligned}$$

For $n = 2$ we obtain all the self dual Einstein and dKP type equations.

Hence we obtain a common structure behind all these integrable system, so there is a consistent and coherent way to describe all these systems. Here we unify all these classes of integrable systems by Gindikin's pencil or bundle of forms and Riemann-Hilbert problem (twistor description). Gindikin introduced these technique to study the geometry of the solution of self dual Einstein equations. Later Takasaki-Takebe [6,7] applied it to dispersionless KP and Toda equations.

This paper is organized as follows:

In section 2 we give some working definitions of Nambu-Poisson geometry. In section 3 we construct our volume preserving integrable systems via Nambu-Poisson geometry. If one carefully analyse these set of equations then one must admit that they bear a close resemblance to the *volume preserving KP equation*, so far nobody knows about this equation. It is known that area preserving KP hierarchy (= dispersionless KP hierarchy) plays an important role in topological minimal models (Landau-Ginzburg description of the A-type minimal models). So we expect volume preserving KP hierarchy may play a big role in low dimensional quantum field theories. Section 4 is dedicated to the twistor construction of these systems.

I would like to end this introduction by expressing some regrets. I apologize to the mathematicians that the presentation of this paper is not regorous, simply because, I want to show the application of the Nambu-Poisson geometry to the large spectrum of readers.

2 Nambu-Poisson Manifolds

In this section we will state some basic results of Nambu-Poisson manifold from the paper of Takhtajan [4].

Let \mathcal{M} denote a smooth finite dimensional manifold and $C^\infty(\mathcal{M})$ the algebra of infinitely differentiable real valued functions on \mathcal{M} . Recall that [4,5] \mathcal{M} is called a Nambu-Poisson manifold if there exists a \mathbb{R} -multi-linear map

$$\{ \cdot, \dots, \cdot \} : [C^\infty(\mathcal{M})]^{\otimes n} \rightarrow C^\infty(\mathcal{M}) \quad (1)$$

called a Nambu bracket of order n such that $\forall f_1, f_2, \dots, f_{2n-1} \in C^\infty(\mathcal{M})$,

$$\{f_1, \dots, f_n\} = (-1)^{\epsilon(\sigma)} \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\}, \quad (2)$$

$$\{f_1 f_2, f_3, \dots, f_{n+1}\} = f_1 \{f_2, f_3, \dots, f_{n+1}\} + \{f_1, f_3, \dots, f_{n+1}\} f_2, \quad (3)$$

and

$$\begin{aligned} & \{ \{f_1, \dots, f_{n-1}, f_n\}, f_{n+1}, \dots, f_{2n-1} \} + \{ f_n, \{f_1, \dots, f_{n-1}, f_{n+1}\}, f_{n+2}, \dots, f_{2n-1} \} \\ & + \dots + \{ f_n, \dots, f_{2n-2}, \{f_1, \dots, f_{n-1}, f_{2n-1}\} \} = \{f_1, \dots, f_{n-1}, \{f_n, \dots, f_{2n-1}\}\}, \end{aligned} \quad (4)$$

where $\sigma \in S_n$ —the symmetric group of n elements—and $\epsilon(\sigma)$ is its parity. Equations (2) and (3) are the standard skew-symmetry and derivation properties found for the ordinary ($n = 2$) Poisson bracket, whereas (4) is a generalization of the Jacobi identity and was called in [2] the fundamental identity. When $n = 3$ this fundamental identity reduces to

$$\begin{aligned} & \{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\} \\ & = \{f_1, f_2, \{f_3, f_4, f_5\}\}. \end{aligned} \quad (5)$$

It is also shown in [2] that Nambu dynamics on a Nambu-Poisson phase space involves $n-1$ so-called Nambu-Hamiltonians $H_1, \dots, H_{n-1} \in C^\infty(\mathcal{M})$ and is governed by the following equations of motion

$$\frac{df}{dt} = \{f, H_1, \dots, H_{n-1}\}, \quad \forall f \in C^\infty(\mathcal{M}). \quad (6)$$

A solution to the Nambu-Hamilton equations of motion produces an evolution operator U_t which by virtue of the fundamental identity preserves the Nambu bracket structure on $C^\infty(M)$.

Definition 2.1 $f \in C^\infty(\mathcal{M})$ is called an integral of motion for the system if it satisfies

$$\{f, H_1, H_2, \dots, H_{n-1}\} = 0.$$

Like the Poisson bivector the Nambu bracket is geometrically realized by a Nambu polyvector $\eta \in \Gamma(\wedge^n TM)$, a section of $\wedge^n TM$, such that

$$\{f_1, \dots, f_n\} = \eta(df_1, \dots, df_n), \quad (7)$$

which in local coordinates (x_1, \dots, x_n) is given by

$$\eta = \eta_{i_1 \dots i_n}(x) \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_n}}, \quad (8)$$

where summation over repeated indices are assumed.

It was stated in [4] that the fundamental identity (4) is equivalent to the following algebraic and differential constraints on the Nambu tensor $\eta_{i_1 \dots i_n}(x)$:

$$S_{ij} + P(S)_{ij} = 0, \quad (9)$$

for all multi-indices $i = \{i_1, \dots, i_n\}$ and $j = \{j_1, \dots, j_n\}$ from the set $\{1, \dots, N\}$, where

$$S_{ij} = \eta_{i_1 \dots i_n} \eta_{j_1 \dots j_n} + \eta_{j_n i_1 i_3 \dots i_n} \eta_{j_1 \dots j_{n-1} i_2} + \dots + \eta_{j_n i_2 \dots i_{n-1} i_1} \eta_{j_1 \dots j_{n-1} i_n} - \eta_{j_n i_2 \dots i_n} \eta_{j_1 \dots j_{n-1} i_1}, \quad (10)$$

and P is the permutation operator which interchanges the indices i_1 and j_1 of $2n$ -tensor S , and

$$\begin{aligned} & \sum_{l=1}^N \left(\eta_{li_2 \dots i_n} \frac{\partial \eta_{j_1 \dots j_n}}{\partial x_l} + \eta_{j_n li_3 \dots i_n} \frac{\partial \eta_{j_1 \dots j_{n-1} i_2}}{\partial x_l} + \dots + \eta_{j_n i_2 \dots i_{n-1} l} \frac{\partial \eta_{j_1 \dots j_{n-1} i_n}}{\partial x_l} \right) \\ &= \sum_{l=1}^N \eta_{j_1 j_2 \dots j_{n-1} l} \frac{\partial \eta_{j_n i_2 \dots i_n}}{\partial x_l}, \end{aligned} \quad (11)$$

for all $i_2, \dots, i_n, j_1, \dots, j_n = 1, \dots, N$.

It was conjectured in [2] that the equation $S_{ij} = 0$ is equivalent to the condition that n -tensor η is decomposable, recently this has been proved by Alekseevsky and author[8], and more recently by Marmo et. al. [9] and many others, so that any decomposable element in

$\wedge^n V$, where V is an N -dimensional vector space over \mathbb{R} , endows V with the structure of a Nambu-Poisson manifold.

Example

Let us illustrate how Nambu-Poisson mechanics works in practise. The example is the motion of a rigid body with a torque about the major axis introduced by Bloch and Marsden [10].

Euler's equation for the rigid body with a single torque u about its major axis is given by

$$\dot{m}_1 = a_1 m_2 m_3 \quad (12)$$

$$\dot{m}_2 = a_2 m_1 m_3 \quad (13)$$

$$\dot{m}_3 = a_3 m_1 m_2 + u \quad (14)$$

where $u = -k m_1 m_2$ is the feedback, $a_1 = \frac{1}{I_2} - \frac{1}{I_3}$, $a_2 = \frac{1}{I_3} - \frac{1}{I_1}$ and $a_3 = \frac{1}{I_1} - \frac{1}{I_2}$. We assume $I_1 < I_2 < I_3$.

These equations can be easily recast into generalized Nambu-Hamiltonian equations of motion:

$$\frac{dm_i}{dt} = \{H_1, H_2, m_i\}, \quad (15)$$

where these equations involve two Hamiltonians and these are

$$H_1 = \frac{1}{2}(a_2 m_1^2 - a_1 m_2^2)$$

$$H_2 = \frac{1}{2}\left(\frac{a_3 - k}{a_1} m_1^2 - m_3^2\right).$$

When

$$a_1 = a_2 = a_3 = 1$$

and $u_3 = 0$, these set of equations reduce to a famous Euler equation or Nahm's equation

$$\frac{dT_i}{dt} = \epsilon_{ijk}[T_j, T_k] \quad i, j, k = 1, 2, 3,$$

where T_i s are $SU(2)$ generators.

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3 Volume preserving integrable systems

In this section we shall follow the approach of Takasaki-Takebe's [6,7] method of studying area preserving diffeomorphic (or sDiff(2)) K.P. equation. In fact, they adopted their method from self dual vacuum Einstein equation theory. In the case of self dual vacuum Einstein equation, and hyperKähler geometry also area preserving diffeomorphism appear, where the spectral variable is merely parameter. But in case of sDiff(2) K.P. equation the situation is different, where Takasaki-Takebe showed that one has to treat λ as a true variable and it enters into the definition of the Poisson bracket. The situation is similar here, onle the Kähler like two form and the associated "Darboux coordinates" is replaced by volume form and the Poisson bracket is replaced by its higher order Poisson bracket called Nambu bracket.

Suppose we consider $L = L(\lambda, p, q)$, $M = M(\lambda, p, q)$ and $N = N(\lambda, p, q)$ are some Laurent series in λ with coefficients are functions of p and q .

Definition 3.1 *The volume preserving integrable hierarchy is defined by*

$$\frac{\partial L}{\partial t_n} = \{B_{1n}, B_{2n}, L\} \quad (16)$$

$$\frac{\partial M}{\partial t_n} = \{B_{1n}, B_{2n}, M\} \quad (17)$$

$$\frac{\partial N}{\partial t_n} = \{B_{1n}, B_{2n}, N\} \quad (18)$$

and

$$\{L, M, N\} = 1, \quad (19)$$

where $B_{1n} : = (L^n)_{n \geq 0}$ and $B_{2n} : = (M^n)_{n \geq 0}$. The first three equations are hierarchy equations and these constitute three flows of the system and the last one shows the volume preservation condition.

Let us now compare our case with the area preserving KP hierarchy. The sdiff(2) KP hierarchy which is given by

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\} \quad \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\} \quad \{\mathcal{L}, \mathcal{M}\} = 1,$$

where \mathcal{L} is a Laurent series in an indeterminant λ of the form

$$\mathcal{L} = \lambda + \sum_{n=1}^{\infty} u_{n+1}(t) \lambda^{-n},$$

$\mathcal{B}_n = (\mathcal{L})_{\geq 0}$. The function \mathcal{M} is called Orlov function and it is defined by

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + \sum_{i=1}^{\infty} v_i \mathcal{L}^{-i-1}$$

where $t_1 = x$.

Remark 3.2 *Our hierarchy has a structure of volume preserving KP hierarchy and instead of one Orlov function we need two Orlov functions Viz. \mathcal{M} and \mathcal{N} .*

Since the flows are commuting, we obtain

Proposition 3.3 *The Lax equation for L , M , N are equivalent to the following equations:*

$$\begin{aligned} & \left\{ \frac{\partial B_{1n}}{\partial t_m}, B_{2n} \right\} + \left\{ B_{1n}, \frac{\partial B_{2n}}{\partial t_m} \right\} + \{ \hat{B}_1, B_{2m} \} \\ & - \left\{ \frac{\partial B_{1m}}{\partial t_n}, B_{2m} \right\} - \left\{ B_{1m}, \frac{\partial B_{2m}}{\partial t_n} \right\} + \{ B_{1m}, \hat{H}_2 \} = 0, \end{aligned}$$

where

$$\hat{H}_1 = \{ B_{1n}, B_{2n}, B_{1m} \}$$

and

$$\hat{H}_2 = \{ B_{1n}, B_{2n}, B_{2m} \}.$$

Proof: Result follows from the compatibility conditions of hierarchy equations and the fundamental identity.

□

Remark 3.4 *In the case ordinary Poisson geometry and $s\text{Diff}(2)$ hierarchy this is a zero curvature equation.*

Let Ω be a three form given by

$$\Omega := \sum_{n=1}^{\infty} dB_{1n} \wedge dB_{2n} \wedge dt_n = d\lambda \wedge dp \wedge dq + \sum_{n=2}^{\infty} dB_{1n} \wedge dB_{2n} \wedge dt_n.$$

From the definition it is clear Ω is closed 3 form. In fact $s\text{Diff}(3)$ structure is clearly exhibited from this structure and the theory is integrable in the sense of nonlinear graviton construction. This is a generalization of nonlinear graviton construction.

Theorem 3.5 *The volume preserving hierarchy is equivalent to the exterior differential equation*

$$\Omega = dL \wedge dM \wedge dN.$$

Proof: We have seen that Ω can be written in two ways. Expanding both sides of the exterior differential equation as linear combinations of $d\lambda \wedge dp \wedge dq$, $d\lambda \wedge dp \wedge dt_n$, $d\lambda \wedge dq \wedge dt_n$ and $dp \wedge dq \wedge dt_n$.

When we pick up the coefficients of $d\lambda \wedge dp \wedge dq$, we obtain the volume preserving condition

$$\{L, M, N\} = 1.$$

When we equate the other coefficients, viz. $d\lambda \wedge dp \wedge dt_n$, $d\lambda \wedge dq \wedge dt_n$ and $dp \wedge dq \wedge dt_n$ we obtain the following identities:

$$\frac{\partial(B_{1n}, B_{2n})}{\partial(\lambda, p)} = \frac{\partial(L, M, N)}{\partial(\lambda, p, t_n)}, \quad (20)$$

$$\frac{\partial(B_{1n}, B_{2n})}{\partial(\lambda, q)} = \frac{\partial(L, M, N)}{\partial(\lambda, q, t_n)} \quad (21)$$

and

$$\frac{\partial(B_{1n}, B_{2n})}{\partial(p, q)} = \frac{\partial(L, M, N)}{\partial(p, q, t_n)} \quad (22)$$

respectively. We multiply the above three equations by $\frac{\partial L}{\partial q}$, $\frac{\partial L}{\partial p}$ and $\frac{\partial L}{\partial \lambda}$ respectively. If we add first and third equations and subtract the second one from them, then after using the volume preserving identity we obtain,

$$\frac{\partial L}{\partial t_n} = \{B_{1n}, B_{2n}, L\}.$$

Similarly we can obtain the other equations also, in that case we will multiply the equations (20),(21) and (22) by $\frac{\partial M}{\partial q}$, $\frac{\partial M}{\partial p}$ and $\frac{\partial M}{\partial \lambda}$ respectively.

□

We can write down the fundamental relation

$$\Omega = dL \wedge dM \wedge dN$$

by

$$d(MdL \wedge dN + \sum_{n=1}^{\infty} B_{1n}dB_{2n} \wedge dt_n) = 0. \quad (23)$$

This implies the existence of one form Q such that

$$dQ = Md(LdN) + \sum_{n=1}^{\infty} B_{1n}d(B_{2n}dt_n). \quad (24)$$

This is an analogue of "Krichever potential" in the volume preserving case. Hence we can say from (24)

$$M = \frac{\partial Q}{\partial(LdN)}|_{B_{2n}, t_n \text{ fixed}} \quad (25)$$

$$B_{1n} = \frac{\partial Q}{\partial(B_{2n}dt_n)}|_{L, N, B_{2m}, t_m (m \neq n) \text{ fixed}}. \quad (26)$$

4 Application to multidimensional integrable systems and Riemann-Hilbert problem

We already stated that our situation is quite similar to nonlinear graviton construction of Penrose [11] for the self dual Einstein equation. This is a generalization of nonlinear graviton constructions.

To the geometer self dual gravity is nothing but Ricci flat Kähler geometry and it is characterized by the underlying symmetry groups $sDiff(2)$, this are called area preserving diffeomorphism group on surfaces. These are the natural generalization of the groups $Diff(S^1)$, diffeomorphism of circle.

It is well known how the area preserving diffeomorphism group appears in the self dual gravity equation. Let us give a very rapid description of this.

Let us start from a complexified metric of the following form

$$\begin{aligned} ds^2 &= \det \begin{pmatrix} e^{11} & e^{12} \\ e^{21} & e^{22} \end{pmatrix} \\ &= e^{11}e^{22} - e^{12}e^{21}, \end{aligned}$$

where e^{ij} are independent one forms. Ricci flatness condition boils down to the closedness of

$$d\Omega^{kl} = 0 \tag{27}$$

of the exterior 2-forms

$$\Omega^{kl} = \frac{1}{2} J_{ij} e^{ik} \wedge e^{jl}, \tag{28}$$

where (J) is the normalized symplectic form, i.e. it is a 2×2 matrix whose entries are 0, 1, -1 and 0 respectively. Then above system of two forms can be recast to

$$\Omega(\lambda) = \frac{1}{2} J_{ij} (e^{i1} + e^{i2}\lambda) \wedge (e^{j1} + e^{j2}\lambda). \tag{29}$$

This satisfies

$$\Omega(\lambda) \wedge \Omega(\lambda) = 0 \tag{30}$$

$$d\Omega(\lambda) = 0, \tag{31}$$

where d stands for total differentiation. These suggest us to introduce a pair of Darboux coordinates

$$\Omega(\lambda) = dP \wedge dQ$$

and these are the sections of the twistor fibration

$$\pi : \mathcal{T} \longrightarrow CP^1,$$

where \mathcal{T} is the curved twistor space introduced by Penrose. Basically each fibre is endowed with a symplectic form and as the base point moves this also deforms and here comes the area preservation.

Two pairs of Darboux coordiantes are related by

$$f(\lambda, P(\lambda), Q(\lambda)) = P'$$

$$g(\lambda, P(\lambda), Q(\lambda)) = Q'$$

and f and g satisfy $\{f, g\} = 1$. The pair (f, g) is called twistor data. Locally f and g (after twisting with λ) yield patching function. Ricci flat Kähler metric is locally encoded in this data. This set up is nothing but the Riemann-Hilbert problem in area preserving diffeomorphism case.

The novelty of this approach is that this twistor construction will work in the higher dimensions too, when there is no twistor projection. The most important example is the electro-vacuum equation, volume preserving diffeomorphism groups in three dimension play a vital role here. This model was first introduced by Flaherty [12] and later Takasaki [2] showed how this works explicitly.

4.1 Gindikin's bundle of forms

We already stated that anti-self dual vacuum equations govern the behaviour of complex 4-metrics of signature $(+, +, +, +)$ whose Ricci curvature is zero and whose Weyl curvature is self dual. These two curvatures are independent of change of coordinates, so in one particular of the equations these metric becomes autometically Kähler and can be expressed in terms of a single scalar function Ω , the Kähler potential. Then curvarure conditions will lead you to Ist Plebensi's Heavenly equation

$$\frac{\partial^2 \Omega}{\partial x \partial \tilde{x}} \frac{\partial^2 \Omega}{\partial y \partial \tilde{y}} - \frac{\partial^2 \Omega}{\partial x \partial \tilde{y}} \frac{\partial^2 \Omega}{\partial y \partial \tilde{x}} = 1 \quad (32)$$

and the corresponding anti-self-dual Ricci flat metric is

$$g(\Omega) = \frac{\partial^2 \Omega}{\partial x^i \partial \tilde{x}^j} dx^i d\tilde{x}^j, \quad \tilde{x}^i = \tilde{x}, \tilde{y}, \quad x^j = x, y.$$

This system is completely integrable and it is an example of a multidimensional integrable system.

Let Ω be the 2-form, given by

$$\Omega = dx \wedge dy + \lambda(\Omega_{x\tilde{x}} dx \wedge d\tilde{x} + \Omega_{x\tilde{y}} dx \wedge d\tilde{y} + \Omega_{y\tilde{x}} dy \wedge d\tilde{x} + \Omega_{y\tilde{y}} dy \wedge d\tilde{y}) + \lambda^2 d\tilde{x} \wedge d\tilde{y}.$$

Lemma 4.1

- (1) $d\Omega = 0$
- (2) $\Omega \wedge \Omega = 0$.

Proof: Since Ω satisfies Plebanski's Heavenly 1st equation, hence (2) is true.

□

A number of multidimensional integrable systems can be written in terms of a 2-form Ω which satisfies the equations (4.1).

Example:

The dispersionless KP hierarchy has a Lax representation with respect to a series of independent ("time") variables $t = (t_1, t_2, \dots)$

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{B_n, \mathcal{L}\}$$

where

$$B_n := (\mathcal{L}^n)_{\geq 0} \quad n = 1, 2, \dots$$

\mathcal{L} is a Laurent series in an indeterminant λ of the form

$$\mathcal{L} = \lambda + \sum_{n=1}^{\infty} u_{n+1}(t) \lambda^{-n},$$

$\{, \}$ is a Poisson bracket in 2D phase space with respect to (λ, x) .

Let us consider

$$\Omega = d\lambda \wedge dx + \sum_{n=2}^{\infty} dB_n \wedge dt_n$$

then $\Omega \wedge \Omega$ is equivalent to the zero curvature condition

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + \{B_n, B_m\} = 0. \quad (33)$$

This is an alternative form of dispersionless KP hierarchy.

All these systems are related to area preserving diffeomorphism group $\text{sDiff}(2)$.

In this paper we are presenting an analogous picture for multidimensional integrable systems related to volume preserving diffeomorphism group.

Proposition 4.2 *The 3-form*

$$\Omega^{(3)} = d\lambda \wedge dp \wedge dq + \sum_{n=2}^{\infty} dB_{1n} \wedge dB_{2n} \wedge dt_n,$$

satisfies

$$\begin{aligned} d\Omega^{(3)} &= 0 \\ \Omega^{(3)} \wedge \Omega^{(3)} &= 0. \end{aligned}$$

☐
$$\begin{aligned} e^1(\tau) &= e^{11}\tau_1^k + \cdots + e^{1k}\tau_2^k, \\ e^2(\tau) &= e^{21}\tau_1^k + \cdots + e^{2k}\tau_2^k, \\ &\vdots \\ e^{2l}(\tau) &= e^{2l1}\tau_1^k + \cdots + e^{2lk}\tau_2^k. \end{aligned}$$
$$\Omega^k(\tau) = e^1(\tau) \wedge e^2(\tau) + \cdots + e^{2l-1}(\tau) \wedge e^{2l}(\tau)$$

- $(\Omega^k)^{l+1} = 0$
- $(\Omega^k)^l \neq 0$
- $d\Omega^k = 0$.

$$g = e^{11}e^{22} - e^{12}e^{21}.$$
$$\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{C}^2,$$

$$g = e^{11}e^{22}e^{33} - e^{11}e^{32}e^{23} + e^{12}e^{31}e^{23} \\ - e^{12}e^{21}e^{33} + e^{13}e^{21}e^{32} - e^{13}e^{31}e^{22}.$$

Remark 4.3 Since 3-form $\Omega^{(3)}$ satisfies

$$d\Omega^{(3)} = 0$$

$$\Omega^{(3)} \wedge \Omega^{(3)} = 0,$$

so it denotes the volume preserving multidimensional integrable systems. These systems can be described by Gindikin's bundle of multi forms, higher dimensional analogue of nonlinear graviton.

4.3 Twistor description of Volume preserving multidimensional integrable systems

The natural question would be to find out the analogous Riemann-Hilbert problem in the volume preserving case. Our situation is very similar to electro-vacuum equation. Let us consider two sets of solutions of hierarchy (L, M, N) and $(\hat{L}, \hat{M}, \hat{N})$ with different analysis. Then there exist an invertible functional relation between these two sets of functions such that it satisfies

$$\begin{aligned}\hat{L} &= f_1(L, M, N), \quad \hat{M} = f_2(L, M, N), \\ \hat{N} &= f_3(L, M, N),\end{aligned}$$

where $f_1 = f_1(\lambda, p, q)$, $f_2 = f_2(\lambda, p, q)$ and $f_3 = f_3(\lambda, p, q)$ are arbitrary holomorphic functions defined in a neighbourhood of $\lambda = \infty$ except at $\lambda = \infty$.

We assume f_1, f_2, f_3 satisfy the canonical Nambu Poisson relation

$$\{f_1, f_2, f_3\} = 1. \quad (34)$$

This is a kind of Riemann-Hilbert problem related to three dimensional diffeomorphisms. In this case sDiff(3) symmetries is clear, in fact sDiff(3) group acts on (f_1, f_2, f_3) , we can lift this action on (L, M, N) and $(\hat{L}, \hat{M}, \hat{N})$ via Riemann-Hilbert factorization.

4.4 Application to hydrodynamic type systems

There are certain kind of integrable systems, called hydrodynamic type, naturally arise in gas dynamics, hydrodynamics, chemical kinematics and may other situations, these are given by

$$u_t^i = v_j^i(u) u_x^j, \quad (35)$$

where $v_j^i(u)$ is an arbitrary $N \times N$ matrix function of

$$u = (u^1, \dots, u^N), \quad u^i = u^i(x, t) \quad i = 1, \dots, N.$$

The Hamiltonian systems of hydrodynamic type systems considered above have the form

$$u_t^i = \{u^i, H\}, \quad (36)$$

where $H = \int h(u)dx$, is a functional of hydrodynamic type. The Poisson bracket of these systems has the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u)\delta_x(x-y) + b_k^{ij}(u)u_x^k\delta(x-y), \quad (37)$$

called Dubrovin-Novikov type Poisson bracket. Dubrovin-Novikov showed if the metric is non degenerate i.e. $\det [g^{ij}] \neq 0$ then (37) yields a Poisson bracket provided $g^{ij}(u)$ is a metric of zero Riemannian Curvature.

A large class of them can be described by the Lax form,

$$\psi_x = zA\psi$$

$$\psi_t = zB\psi.$$

From the compatibility condition we obtain,

$$A_t = B_x$$

$$AB = BA.$$

These set of equations can be easily recasted to

$$d\Omega = 0$$

$$\Omega \wedge \Omega = 0,$$

where Ω is a closed one form.

Hence has also a twistorial description. The upshot of this section is that Gindikin pencil of forms can be applied to large number of classes integrable systems, including the volume preserving integrable systems we propose here.

5 Conclusion

In this paper we have shown that there are large class of integrable systems can be obtained from Nambu-Poisson mechanics. They belong to the same family of self dual Einstein or dispersionless KP type equations. In fact they are related to what should be called volume preserving KP system. In some sense these integrable systems are higher dimensional generalization of self dual Einstein equation. Hence these systems are describable via Gindikin's bundle of forms, or twistor method.

In fact we have pointed out in this paper that Nambu-Poisson manifold is an useful tool to study volume preserving integrable systems. Recently in membrane theory physicists [14] have found M-algebra from M-brane, these are related to Nambu-Poisson mechanics.

There are certain problems we have not discussed in this paper, viz. the quantization of these volume preserving generalized multidimensional integrable systems. Presumably method of star product quantization [15,16] would be the best way quantize these systems and instead of binary star product we need triple star product [17,18]. Thus it is important to see whether Moyal-Nambu bracket can be expressed in terms of Fedosov's triple star product.

6 References

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